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INVERSE STURM–LIOUVILLE PROBLEMS USING FINITE NUMBER OF TRANSMISSION CONDITIONS

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Abstract. In this paper we consider the inverse Sturm–Liouville problems with a finite number of discontinuities at interior points. We develop the Hochestadt's result based on transformation operator for Sturm–Liouville problem with finite number of transmission conditions. Furthermore, we establish a formula for $q(x) - \tilde{q}(x)$ in the finite interval, where $\tilde{q}(x)$ and $q(x)$ are analogous potential functions with different operators.

Keywords: Inverse Sturm–Liouville problem, internal discontinuities, finite number of transmission conditions, Green's function.

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1 Introduction

The method of separation of variables for solving PDEs with discontinuous boundary conditions naturally led to ODE with discontinuities inside of the interval which often appear in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. For example the mathematical formulation of a large variety of technical and physical problem led to inverse problems such as identifying the density of the thing from data collected from the sets of frequencies of oscillations of the string with barrier.

Sturm–Liouville problems with transmission conditions at interior points arise in a variety of applications in engineering and we refers to Amirov (2006) for a nice discussion and further information. The inverse spectral Sturm–Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions. Inverse problems with the discontinuities conditions inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. As a rule, such problems are related to discontinuous and non-smooth properties of a medium (Anderssen, 1977; Freiling et al., 2001, 2010; Hald, 1984; Krueger, 1982).

We refer to the somewhat complementary surveys in inverse Sturm–Liouville problems with discontinuous conditions in Amirov (2006), Freiling et al. (2010), Hald (1984), Koyunbakan (2008, 2011), Shahriari (2014, 2017, 2021), Shahriari et al. (2012), Shieh et al. (2008), Willis (1985), Yang et al. (2009), Yu-Ping (2011), and Yurko (2000). In this manuscript, we generalize the results of Hochstadt (1973), refining the approach of Levinson (1949) to show that precisely how much q has freedom where the μ_n and all but finitely many of the λ_n are specified. Note that the eigenvalues μ_n is obtained with replacing H by $\mathfrak{H}$ in (2). The similar papers for Hochstadt's result are in Binding et al. (2002), Koyunbakan (2008, 2011), Shahriari (2014, 2021).

2 The Hilbert space formulation and properties of the spectrum

We consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y \tag{1}$$

subject to the Robin boundary conditions

$$\begin{aligned} L_1(y) &:= y'(0) + h\, y(0) = 0, \\ L_2(y) &:= y'(\pi) + H\, y(\pi) = 0, \end{aligned} \tag{2}$$

with transmission (discontinuous) conditions

$$\begin{aligned} U_i(y) &:= y(d_i+0) - a_i y(d_i-0) = 0, \\ V_i(y) &:= y'(d_i+0) - b_i y'(d_i-0) - c_i y(d_i-0) = 0, \end{aligned} \tag{3}$$

where $q(x)$ is real-valued function in $L^2[0,\pi]$. Also we assume that h, H and a_i, b_i, c_i d_i, $i = 1, 2, \ldots, m-1$ (with $m \geq 2$) are real numbers, satisfying $a_i b_i > 0$, $d_0 = 0 < d_1 < d_2 < \ldots < d_{m-1} < d_m = \pi$. For simplicity we use the notation $L = L(q(x); h; H; d_i)$, for the problem (1)–(3). To obtain a self-adjoint operator we define the following weight function

$$w(x) = \begin{cases} 1, & 0 \leq x < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < x < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < x \leq \pi. \end{cases}$$

Now our Hilbert space will be $\mathcal{H} := L_2((0,\pi); w)$ associated with the weighted inner product

$$\langle f, g \rangle_{\mathcal{H}} := \int_0^\pi f \overline{g} w.$$

The corresponding norm will be denoted by $\|f\|_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}^{1/2}$. In this Hilbert space we construct the operator

$$A : \mathcal{H} \to \mathcal{H}$$

with domain

$$\operatorname{dom}(A) = \left\{ f \in \mathcal{H} \;\middle|\; \begin{array}{c} f, f' \in AC\big(\cup_0^{m-1} (d_i, d_{i+1})\big), \\ \ell f \in L^2(0,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\}$$

by

$$Af = \ell f \quad \text{with } f \in \operatorname{dom}(A).$$

Throughout this paper $AC\big(\cup_0^{m-1} (d_i, d_{i+1})\big)$ denotes the set of all functions whose restriction to (d_i, d_{i+1}) is absolutely continuous for all $i = 0, \ldots, m-1$. In particular, those functions will have limits at the boundary points d_i (see Shahriari et al. (2012)).
From the linear differential equations we obtain that the modified Wronskian

$$W(u,v) = w(x)\big(u(x)v'(x) - u'(x)v(x)\big)$$

is constant on $x \in [0, d_1) \cup_1^{m-2} (d_i, d_i+1) \cup (d_{m-1}, \pi]$ for two solutions $\ell u = \lambda u$, $\ell v = \lambda v$ satisfying the transmission conditions (3). Using the formula for the modified Wronskian $W(u,v)(x) = W(u,v)(x_0)$, for $x_0 \in [0, d_1) \cup_1^{m-2} (d_i, d_{i+1}) \cup (d_{m-1}, \pi]$. So, $W(u,v)$ does not depend on x.

Lemma 1. *The operator A is self-adjoint.*

Proof. By employing twice the integration by part, one can write

$$\langle Au, v\rangle_{\mathcal{H}} = W(u,v)\big|_{x=\pi} - W(u,v)\big|_{x=0} + \langle u, Av\rangle_{\mathcal{H}}.$$

It follows from the conditions (2) and (3),

$$W(u,v)\big|_{x=\pi} - W(u,v)\big|_{x=0} = 0.$$

Therefore A is self-adjoint operator on $L_2((0,\pi);w)$. □

In particular, the eigenvalues of A, and hence of L, are real and simple. To see that they are simple it suffices to observe that the associated Cauchy problem (1) and (3) subject to the initial conditions $f(x_0 \pm 0) = f_0$, $f'(x_0 \pm 0) = f_1$ (with $x_0 \in [0,\pi]$) has a unique solution.
For any function $f \in \operatorname{dom}(A)$, we will denote by f_j, $1 \le j \le m$, the restriction of f to the subinterval (d_{j-1}, d_j). Moreover, we will set $f_j(d_{j-1}) = f(d_{j-1}+0)$ and $f_j(d_j) = f(d_j - 0)$.

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (1) under the initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = -h, \tag{4}$$

and

$$\psi(\pi,\lambda) = 1, \quad \psi'(\pi,\lambda) = -H,$$

as well as the jump conditions (3), respectively. Moreover, we set

$$\Delta(\lambda) := W(\varphi(\lambda), \psi(\lambda)) = L_1(\psi(\lambda)) = -w(\pi) L_2(\varphi(\lambda)).$$

Then $\Delta(\lambda)$ is an entire function whose roots λ_n coincide with the eigenvalues of L.

Theorem 1. *(Shahriari et al., 2012, Thm. 3.1) Let $\lambda = \rho^2$ and $\tau := \operatorname{Im}\rho$. For equation (1) with boundary conditions (2) and jump conditions (3) as $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$\varphi(x,\lambda) = \begin{cases} \cos\rho x + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & 0 \le x < d_1, \\ \alpha_1 \cos\rho x + \alpha_1' \cos\rho(x - 2d_1) + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & d_1 < x < d_2, \\ \alpha_1\alpha_2 \cos\rho x + \alpha_1'\alpha_2 \cos\rho(x-2d_1) + \alpha_1\alpha_2' \cos\rho(x-2d_2) & \\ \quad +\alpha_1'\alpha_2' \cos\rho(x + 2d_1 - 2d_2) + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & d_2 < x < d_3, \\ \quad \vdots & \\ \alpha_1\alpha_2 ... \alpha_{m-1} \cos\rho x + & \\ \quad +\alpha_1'\alpha_2...\alpha_{m-1} \cos\rho(x - 2d_1) + \cdots & \\ \quad +\alpha_1\alpha_2...\alpha_{m-1}' \cos\rho(x - 2d_{m-1}) + & \\ \quad +\alpha_1'\alpha_2'\alpha_3...\alpha_{m-1} \cos\rho(x + 2d_1 - 2d_2) + \cdots & \\ \quad +\alpha_1...\alpha_i'...\alpha_j'...\alpha_{m-1} \cos\rho(x + 2d_i - 2d_j) & \\ \quad +\alpha_1...\alpha_i'...\alpha_j'...\alpha_k'...\alpha_{m-1} \cos\rho(x - 2d_i + 2d_j - 2d_k) + \cdots & \\ \quad +\alpha_1'\alpha_2'...\alpha_{m-1}' \cos\rho(x + 2(-1)^{m-1}d_1 + 2(-1)^{m-2}d_2 + \cdots - 2d_{m-1}) & \\ \quad +O\left(\frac{\exp(|\tau|x)}{\rho}\right), & d_{m-1} < x \le \pi, \end{cases}$$

and

$$\varphi'(x,\lambda)=\begin{cases}\rho[-\sin\rho x]+O(\exp(|\tau|x)), \qquad 0\le x<d_1,\\ \rho[-\alpha_1\sin\rho x-\alpha_1'\sin\rho(x-2d_1)]+O(\exp(|\tau|x)), \qquad d_1<x<d_2,\\ \rho[-\alpha_1\alpha_2\sin\rho x-\alpha_1'\alpha_2\sin\rho(x-2d_1)-\\ \qquad -\alpha_1\alpha_2'\sin\rho(x-2d_2)-\alpha_1'\alpha_2'\sin\rho(x+2d_1-2d_2)]\\ \qquad +O(\exp(|\tau|x)), \qquad d_2<x<d_3,\\ \vdots\\ \rho[-\alpha_1\alpha_2...\alpha_{m-1}\sin\rho x-\alpha_1'\alpha_2...\alpha_{m-1}\sin\rho(x-2d_1)-\cdots-\alpha_1\alpha_2...\alpha_{m-1}'\\ \quad \sin\rho(x-2d_{m-1})-\alpha_1'\alpha_2'\alpha_3...\alpha_{m-1}\sin\rho(x+2d_1-2d_2)-\cdots\\ \quad -\alpha_1...\alpha_i'...\alpha_j'...\alpha_{m-1}\sin\rho(x+2d_i-2d_j)\\ \quad -\alpha_1...\alpha_i'...\alpha_j'...\alpha_k'...\alpha_{m-1}\sin\rho(x-2d_i+2d_j-2d_k)+\cdots\\ \quad -\alpha_1'\alpha_2'...\alpha_{m-1}'\sin\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})]\\ \quad +O(\exp(|\tau|x)), \qquad d_{m-1}<x\le\pi,\end{cases}$$

where

$$\alpha_i=\frac{a_i+b_i}{2} \text{ and } \alpha_i'=\frac{a_i-b_i}{2},$$

for $i=1,2,...,m-1$. *The characteristic function satisfies*

$$\begin{aligned}\Delta(\lambda)=&\rho w(\pi)\big[\alpha_1\alpha_2...\alpha_{m-1}\sin\rho\pi+\alpha_1'\alpha_2...\alpha_{m-1}\sin\rho(\pi-2d_1)+\cdots\\ &+\alpha_1\alpha_2...\alpha_{m-1}'\sin\rho(\pi-2d_{m-1})+\alpha_1'\alpha_2'\alpha_3...\alpha_{m-1}\sin\rho(\pi+2d_1-2d_2)\\ &+\cdots+\alpha_1...\alpha_i'...\alpha_j'...\alpha_{m-1}\sin\rho(\pi+2d_i-2d_j)\\ &+\alpha_1...\alpha_i'...\alpha_j'...\alpha_k'...\alpha_{m-1}\sin\rho(\pi-2d_i+2d_j-2d_k)+\cdots\\ &+\alpha_1'\alpha_2'...\alpha_{m-1}'\sin\rho(\pi+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})\big]\\ &+O(\exp(|\tau|\pi)).\end{aligned}$$

From the above theorem it follows that

$$|\varphi^{(\nu)}(x,\lambda)|=O(|\rho|^\nu\exp(|\tau|x)), \quad 0\le x\le\pi,\ \nu=0,1. \tag{5}$$

By changing x to $\pi-x$ one can obtain the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. In particular,

$$|\psi^{(\nu)}(x,\lambda)|=O(|\rho|^\nu\exp(|\tau|(\pi-x))), \quad 0\le x\le\pi,\ \nu=0,1.$$

As a consequence of Valiron's theorem (Levin, 1996, Thm. 13.4) and theorem (Shahriari et al., 2012, Thm. 3.2) we obtain:

Theorem 2. *The eigenvalues* $\lambda_n=\rho_n^2$ *of the boundary value problem* L *satisfy*

$$\rho_n=n+o(n)$$

as $n\to\infty$.

Moreover, the eigenfunctions $\varphi_i(x,\lambda_n)$ and $\psi_i(x,\lambda_n)$ associated with a certain eigenvalue λ_n, satisfy the relation

$$\psi_i(x,\lambda_n)=\beta_n\varphi_i(x,\lambda_n),$$

where, by (4),

$$\beta_n=\psi(0,\lambda_n).$$

We also define the norming constant by

$$\gamma_n:=\|\varphi(x,\lambda_n)\|^2_{\mathcal{H}}. \tag{6}$$

Then it is straightforward to verify:

Lemma 2. *(i) All zeros λ_n of $\Delta(\lambda)$ are simple and the derivative is given by*

$$\dot{\Delta}(\lambda_n) = -\gamma_n\beta_n.$$

(ii) If $\varphi(x,\lambda_n)$ is the eigenfunction corresponding to eigenvalues λ_n, then

$$\gamma_n = \|\varphi(x,\lambda_n)\|^2_{\mathcal{H}} = \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right],$$

where

$$\mu(\rho_n; d_i; a_i; b_i) =$$
$$\begin{cases} \frac{\pi}{2}, \qquad \text{for} \quad m=1, \\[2mm] \frac{d_1}{2} + \frac{\pi-d_1}{2a_1b_1}\big(\alpha_1^2 + \alpha'^2_1 + \alpha_1\alpha'_1\cos\rho_n(2d_1)\big), \qquad \text{for} \quad m=2, \\[2mm] \frac{d_1}{2} + \frac{d_2-d_1}{2a_1b_1}(\alpha_1^2 + \alpha'^2_1 + \alpha_1\alpha'_1\cos\rho_n(2d_1)) \\ \quad + \frac{\pi-d_2}{2a_1b_1a_2b_2}\big(\alpha_1^2\alpha_2^2 + \alpha'^2_1\alpha_1^2 + \alpha'^2_1\alpha'^2_2 + 2\alpha_1\alpha_2\alpha'_1\alpha_2\cos\rho_n(2d_1) \\ \quad + \alpha_1^2\alpha_2\alpha'_2\cos\rho_n(2d_2) + 2\alpha_1\alpha_2\alpha'_1\alpha'_2\cos\rho_n(2d_2-2d_1) \\ \quad + \alpha'^2_1\alpha_2\alpha'_2\cos\rho_n(2d_2-4d_1)\big), \qquad \text{for } m=3, \\ \quad \vdots \\ \frac{d_1}{2} + \frac{d_2-d_1}{2a_1b_1}(\alpha_1^2 + \alpha'^2_1 + \alpha_1\alpha'_1\cos\rho_n(2d_1)) \\ \quad + \frac{d_3-d_2}{2a_1b_1a_2b_2}\big(\alpha_1^2\alpha_2^2 + \alpha'^2_1\alpha_1^2 + \alpha'^2_1\alpha'^2_2 + 2\alpha_1\alpha_2\alpha'_1\alpha_2\cos\rho_n(2d_1) \\ \quad + 2\alpha_1\alpha_2\alpha'_1\alpha'_2\cos\rho_n(2d_2-2d_1) + \alpha'^2_1\alpha_2\alpha'_2\cos\rho_n(2d_2-4d_1) \\ \quad + \alpha_1^2\alpha_2\alpha'_2\cos\rho_n(2d_2)\big) + \cdots \\ \quad + \frac{1}{2a_1b_1\ldots a_{m-1}b_{m-1}}\big(\alpha_1^2\alpha_2^2\ldots\alpha_{m-1}^2 + \alpha'^2_1\alpha_2^2\ldots\alpha_{m-1} + \cdots + \alpha_1^2\alpha_2^2\ldots\alpha'^2_{m-1} \\ \quad + \cdots + \alpha'^2_1\alpha'^2_2\ldots\alpha'^2_{m-1} + \alpha_1\alpha'_1\alpha_2^2\ldots\alpha_{m-1}^2\cos\rho_n(2d_1) \\ \quad + \alpha_1^2\alpha_2\alpha'_2\alpha_3^2\ldots\alpha_{m-1}^2\cos\rho_n(2d_2) \\ \quad + \alpha_1^2\alpha_2^2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2d_{m-1}) + \cdots \\ \quad + \alpha'^2_1\alpha_2\alpha'_2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2[(-1)^{m-1}+1]d_1 + 2(-1)^{m-2}d_2 + \cdots - 2d_{m-1}) \\ \quad + \alpha_1\alpha'_1\alpha'_2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2(-1)^{m-1}d_1 + 2[(-1)^{m-2}+1]d_2 + \cdots - 2d_{m-1}) \\ \quad + \cdots + \alpha_1\alpha'_1{\alpha'_2}^2\ldots{\alpha'_{m-1}}^2\cos\rho_n(2d_1) \\ \quad + \cdots + \alpha'^2_1{\alpha'_2}^2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2d_{m-1})\big), \qquad \text{for } m>3. \end{cases}$$

By using the similar proof of theorem ((Freiling et al., 2001, Thm:1.2.1)) we get:

Theorem 3. *The system of eigenfunctions $\{\varphi_n(x) := \varphi(x,\lambda_n)\}_{n\geq 0}$ of the boundary value problem A is complete in $L_2((0,\pi),w)$.*

3 Transformation operator

In this section, we investigate the transformation operator for two operators L and $\tilde{L}$, where $\tilde{L} = L(\tilde{q}(x); h; H; d_i)$ and $\mathrm{dom}(\tilde{A})$ defined by an analogous manner with ℓ replaced by $\tilde{\ell}$, where $\tilde{\ell}y := -y'' + \tilde{q}(x)y$.

Let $L(q(x); h; \mathfrak{H}; d_i)$ be another eigenvalue problem such that by assuming $H - \mathfrak{H} \neq 0$. Suppose that $\theta(x,\lambda)$ is the solution of (1) satisfying the initial conditions $\theta(\pi,\lambda) = 1$, $\theta'(\pi,\lambda) = -\mathfrak{H}$ and the transmission conditions (3). It is clear that $\phi(\lambda) := -w(\pi)L_2(\varphi(\lambda))$ is the characteristic function of $L(q(x); h; \mathfrak{H}; d_i)$ and the zeros of $\phi(\lambda)$ are eigenvalues of $L(q(x); h; \mathfrak{H}; d_i)$, say $\{\mu_n\}_{n=1}^{\infty}$, are real and simple. Define $\tilde{\phi}(\lambda)$ by an analogous manner.

Lemma 3. *If $L(q(x);h;\mathfrak{H};d_i)$ and $L(\tilde{q}(x);h;\mathfrak{H};d_i)$ have the same eigenvalues, i.e.($\mu_n = \tilde{\mu}_n$) and $w(\pi) = \tilde{w}(\pi)$ then $\phi = \tilde{\phi}$.*

Proof. From Levin (1996) it follows that ϕ and $\tilde{\phi}$ are entire functions of order $1/2$, consequently by using Hadamard's factorization theorem, we have

$$\phi(\lambda) = C \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right)$$

and

$$\tilde{\phi}(\lambda) = \tilde{C} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\tilde{\mu}_n}\right).$$

Define $F(\lambda) := \frac{\phi(\lambda)}{\tilde{\phi}(\lambda)}$. From the assumptions we see that $\phi(\mu_n) = \tilde{\phi}(\mu_n) = 0$, therefore $F(\lambda)$ is an entire function in λ. By using the asymptotic form of $\phi(\lambda)$ and $\tilde{\phi}(\lambda)$ from (5) with H replaced by $\mathfrak{H}$, we get

$$F(\lambda) = 1 + o(1).$$

So by applying Liouville's Theorem, we get

$$\phi(\lambda) = \tilde{\phi}(\lambda).$$

□

Suppose that the functions $\tilde{\varphi}(x,\lambda)$ and $\tilde{\psi}(x,\lambda)$ are solutions of

$$\tilde{\ell}y = -y'' + \tilde{q}y = \lambda y$$

under the initial conditions

$$\tilde{\varphi}(0,\lambda) = 1, \quad \tilde{\varphi}'(0,\lambda) = -h,$$

and

$$\tilde{\psi}(\pi,\lambda) = 1, \quad \tilde{\psi}'(\pi,\lambda) = -H,$$

as well as the jump conditions (3), respectively. So we get

$$\tilde{\psi}_i(x,\tilde{\lambda}_n) = \tilde{\beta}_n \tilde{\varphi}_i(x,\tilde{\lambda}_n),$$

where

$$\tilde{\beta}_n = \tilde{\psi}(0,\tilde{\lambda}_n),$$

where $\tilde{\varphi}_i(x,\tilde{\lambda}_n)$ and $\tilde{\psi}_i(x,\tilde{\lambda}_n)$ be eigenfunctions of $L(\tilde{q}(x);h;H;d_i)$ corresponding to the eigenvalue $\tilde{\lambda}_n$.

Lemma 4. *Let $\Lambda_0 \subset \mathbb{N}$ be a finite set and $\Lambda = \mathbb{N}\backslash\Lambda_0$. If $L(q(x);h;\mathfrak{H};d_i)$, $L(\tilde{q}(x);h;\mathfrak{H};d_i)$ have the same eigenvalues and, as well as, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of $L(q(x);h;H;d_i)$ and $L(\tilde{q}(x);h;H;d_i)$, then $\beta_n = \tilde{\beta}_n$ for all $n \in \Lambda$.*

Proof. From definition of φ, θ and ψ it follows that

$$\begin{cases} W(\varphi_n(x),\psi_n(x)) = -w(\pi)(\varphi_n'(\pi) + H\varphi_n(\pi)) = 0, \\ W(\varphi_n(x),\theta_n(x)) = -w(\pi)(\varphi_n'(\pi) + \mathfrak{H}\varphi_n(\pi)) = \phi(\lambda_n). \end{cases}$$

The above linear system has a unique solution

$$\varphi_n(\pi) = \frac{\phi(\lambda_n)}{h - \mathfrak{H}}, \quad \varphi_n'(\pi) = -H\varphi_n(\pi).$$

Similarly we obtain

$$\tilde{\varphi}_n(\pi) = \frac{\tilde{\phi}(\tilde{\lambda}_n)}{h - \mathfrak{H}}, \quad \tilde{\varphi}'_n(\pi) = -H\tilde{\varphi}_n(\pi).$$

From $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and Lemma 3, we have $\phi(\lambda_n) = \tilde{\phi}(\lambda_n)$. So we get

$$\beta_n = \tilde{\beta}_n$$

for all $n \in \Lambda$. □

Let

$$U := \mathrm{dom}(A) \ominus \{\varphi_n : n \in \Lambda_0\},$$
$$\tilde{U} := \mathrm{dom}(\tilde{A}) \ominus \{\tilde{\varphi}_n : n \in \Lambda_0\}.$$

Define the transformation operator $T : U \to \tilde{U}$ by

$$T\varphi_n = \tilde{\varphi}_n,$$

for all $n \in \Lambda$. Note that by $\mathrm{dom}\,(A) \ominus \{\varphi_n : n \in \Lambda_0\}$ we mean $\mathrm{dom}\,(A)$ contains all of $\{\varphi_n\}_{n=1}^{\infty}$ except $\{\varphi_n\}_{n\in\Lambda_0}$.

Lemma 5. *The operator $T : U \to \tilde{U}$ is bounded.*

Proof. From Lemma 2 we see that

$$\tilde{\gamma}_n = \parallel \tilde{\varphi}_n \parallel^2 = \int_0^\pi \tilde{\varphi}_n^2(t)w(t)dt = \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right] \tag{7}$$

and

$$\gamma_n = \parallel \varphi_n \parallel^2 = \int_0^\pi \varphi_n^2(t)w(t)dt = \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right] \tag{8}$$

for all $n \in \Lambda$. Thus by (7) and (8) we get

$$\frac{\parallel T\varphi_n \parallel^2}{\parallel \varphi_n \parallel^2} = \frac{\parallel \tilde{\varphi}_n \parallel^2}{\parallel \varphi_n \parallel^2} = \frac{1 + O\left(\frac{1}{n}\right)}{1 + O\left(\frac{1}{n}\right)} = 1 + O\left(\frac{1}{n}\right).$$

□

Lemma 6. *For the operator T, the relation $(\lambda - \tilde{A})T(\lambda - A)^{-1} = T$ holds.*

Proof. Assume that $f \in U$. We can extend f in term of the set $\{\varphi_n\}$,

$$f(x) = \sum_{\Lambda} f_n\varphi_n(x),$$

where

$$f_n = \frac{\langle f, \varphi_n\rangle_{\mathcal{H}}}{\langle \varphi_n, \varphi_n\rangle_{\mathcal{H}}}, \qquad \text{for } n \in \Lambda.$$

Assume that λ is in the complex plane and is not an eigenvalue of $L(q(x); h; H; d_i)$, then the operator $(\lambda - A)^{-1}$ exists and bounded. So we can write as the following form

$$(\lambda - A)^{-1}f(x) = \sum_{\Lambda} \frac{f_n\varphi_n(x)}{\lambda - \lambda_n}.$$

We now get

$$T(\lambda - A)^{-1}f(x) = \sum_{\Lambda} \frac{f_n\tilde{\varphi}_n(x)}{\lambda - \lambda_n}$$

and

$$(\lambda - \tilde{A})T(\lambda - A)^{-1}f(x) = \sum_{\Lambda} f_n\tilde{\varphi}_n(x).$$

Then we have

$$(\lambda - \tilde{A})T(\lambda - A)^{-1} = T.$$

□

4 Inverse problem

In this section, we examine a different representation for T, in a general case when there are m discontinuous conditions. We generalize the well–known result of Hochstadt Hochstadt (1973). Denote

$$G(x,y;\lambda) := \begin{cases} \frac{\varphi(x,\lambda)\psi(y,\lambda)}{\Delta(\lambda)}, & 0 < x < y < \pi, \\ \frac{\varphi(y,\lambda)\psi(x,\lambda)}{\Delta(\lambda)}, & 0 < y < x < \pi, \end{cases} \tag{9}$$

where $x, y \neq d_i$. For simplicity we can write

$$G(x,y;\lambda) = \frac{\varphi(x <)\psi(x >)}{\Delta(\lambda)}$$

where $x <:= \min\{x,y\}$ and $x >:= \max\{x,y\}$ and consider the function

$$Y(x,\lambda) = \int_0^\pi G(x,y,\lambda)f(y)w(y)dy.$$

The function $G(x,y;\lambda)$ is called the Green's function for A. $G(x,y;\lambda)$ is the kernel of the inverse operator for the Sturm-Liouville problem, i.e. $Y(x,\lambda)$ is the solution of the boundary value problem

$$\ell Y - \lambda Y = f(x), \quad U(Y) = V(Y) = 0; \tag{10}$$

and the jump conditions (3), this is easily verified by differentiation. Let C_n be a sequence of circles about the origin intersecting the positive λ-axis between λ_n and λ_{n+1}. By using Eq. (9), we get

$$\lim_{n\to\infty} \int_{C_n} \frac{G(x,y;\mu)}{\lambda-\mu} d\mu = 0, \quad \lambda \in \operatorname{int} C_n. \tag{11}$$

From residue integration, it follows that

$$\frac{1}{2\pi i}\int_{C_n} \frac{G(x,y;\mu)}{\lambda-\mu} d\mu = -G(x,y;\lambda) + \sum_{k=0}^{n} \frac{\varphi_k(x <)\psi_k(x >)}{\dot{\Delta}(\lambda_k)(\lambda-\lambda_k)}. \tag{12}$$

By applying Mittag-Leffler expansion for $G(x,y,\lambda)$ and using (11) and (12) we obtain

$$G(x,y;\lambda) = \sum_{k=0}^{\infty} \frac{\varphi_k(x <)\psi_k(x >)}{\dot{\Delta}(\lambda_k)(\lambda-\lambda_k)}, \tag{13}$$

where φ_k, ψ_k are eigenfunctions corresponding to the eigenvalues λ_k.

Theorem 4 (Main Theorem). *If $L(q(x);h;\mathcal{H};d_i)$, $\tilde{L}(\tilde{q}(x);h;\mathcal{H};d_i)$ have the same spectrum and $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, then*

$$q - \tilde{q} = \sum_{\Lambda_0} (\tilde{y}_n\varphi_n)' w,$$

a.e. on $[0,d_1) \cup_{i=1}^{m-3} (d_i, d_{i+1}) \cup (d_{m-1}, \pi]$, where $\tilde{y}_n$ and φ_n are suitable solutions of $\tilde{\ell}y = \lambda_n y$ and $\ell y = \lambda_n y$, respectively.

Proof. From (10) and (13) we obtain

$$\begin{aligned}(\lambda - A)^{-1} f(x) &= \int_0^{\pi} G(x,y;\lambda) f(y) w(y) dy \\ &= \frac{\psi(x)\int_0^x \varphi(y) f(y) w(y) dy + \varphi(x) \int_x^{\pi} \psi(y) f(y) w(y) dy}{\Delta(\lambda)} \\ &= \sum_{\Lambda} \frac{\psi_n(x)\int_0^x \varphi_n(y) f(y) w(y) dy + \varphi_n(x) \int_x^{\pi} \psi_n(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ &= \sum_{\Lambda} \frac{k_n \varphi_n(x) \int_0^{\pi} \varphi_n(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}\end{aligned} \tag{14}$$

for $f \in U$. By applying T to (14) we get

$$T(\lambda - A)^{-1} f(x) = \sum_{\Lambda} \frac{k_n \tilde{\varphi}_n(x) \int_0^{\pi} \varphi_n(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}. \tag{15}$$

Define

$$g(x) := \frac{\tilde{\psi}(x)\int_0^x \varphi(y) f(y) w(y) dy + \tilde{\varphi}(x) \int_x^{\pi} \psi(y) f(y) w(y) dy}{\Delta(\lambda)}.$$

By applying the Mittag-Leffler expansion for $g(x)$, we have

$$\begin{aligned} g(x) =& \sum_{\Lambda_0} \frac{\tilde{u}_n(x)\int_0^x \varphi_n(y) f(y) w(y) dy + \tilde{z}_n(x) \int_x^{\pi} \psi_n(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ &+ \sum_{\Lambda} \frac{\tilde{\psi}_n(x)\int_0^x \varphi_n(y) f(y) w(y) dy + \tilde{\varphi}_n(x) \int_x^{\pi} \psi_{1n}(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}. \end{aligned} \tag{16}$$

The second term of the expression (16) is equal to $T(\lambda - A)^{-1} f$ in (15). The functions $\tilde{u}_n(x)$ and $\tilde{z}_n(x)$ represents $\tilde{\psi}(x, \lambda_n)$ and $\tilde{\varphi}(x, \lambda_n)$, respectively. Hence

$$\begin{aligned} (\lambda - \tilde{A})^{-1} T f(x) &= g(x) \\ &- \sum_{\Lambda_0} \frac{\tilde{u}_n(x)\int_0^x \varphi_n(y) f(y) w(y) dy + \tilde{z}_n(x) \int_x^{\pi} \psi_n(y) f(y) w(y) dy}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}. \end{aligned} \tag{17}$$

The right and left-hand side of (17) are in the domain of $(\lambda - \tilde{A})$. By using a simple calculation, we get

$$T f(x) = f(x) - \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_n(x) \int_0^x \varphi_n(y) f(y) w(y) dy,$$

where

$$\frac{1}{2} \tilde{y}_n(x) = \frac{\tilde{u}_n(x) - \beta_n \tilde{z}_n(x)}{\dot{\Delta}(\lambda_n)}.$$

From Lemma 6 and Eq. (17) it follows that $\tilde{A} T f = T A f$. For $f = \varphi_n(x)$ we have

$$\begin{aligned} \tilde{A} T \varphi_n =& \tilde{A} \left(\varphi_n - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_m \int_0^x \varphi_m \varphi_n w \right) \\ =& \tilde{A} \varphi_n - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{A} \tilde{y}_m \int_0^x \varphi_m \varphi_n w \\ & - \frac{1}{2} \sum_{m \in \Lambda_0} 2 \tilde{y}_m' (\varphi_m \varphi_n) w - \frac{1}{2} \sum_{m \in \Lambda_0} \tilde{y}_m (\varphi_m \varphi_n w)'. \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
TA\varphi_n &= T(-\varphi_n'' + q\varphi_n) \\
&= A\varphi_n - \frac{1}{2}\sum_{m\in\Lambda_0} \tilde{y}_m \int_0^x \varphi_m\varphi_n w - \frac{1}{2}\sum_{m\in\Lambda_0} \tilde{y}_m(\varphi_n\varphi_{\varphi_m}' - \varphi_m\varphi_n')w.
\end{aligned} \tag{19}
$$

From (18) and (19) we deduce that

$$
q - \tilde{q} = \sum_{\Lambda_0} (\tilde{y}_m\varphi_m)'w.
$$

If Λ_0 is empty, then T is a unitary operator and $A = \tilde{A}$. Hence $q = \tilde{q}$. □

Theorem 5. *Let λ_n and φ_n denote the eigenvalues and eigenfunctions of $L(q(x), h, H, d_i)$ where φ_n, is normalized by $\varphi_n(0) = 1$, $\varphi_n'(0) = -h$. Define $\tilde{\lambda}_n$ and $\tilde{\varphi}_n$ in an analogous manner but with A replaced by $\tilde{A}$. Suppose that $\lambda_n = \tilde{\lambda}_n$ and $\gamma_n = \tilde{\gamma}_n$, where γ_n is defined in (6), for all $n \in \Lambda_0$, then*

$$
q = \tilde{q}.
$$

Proof. From Lemma 3 applied to $L(q(x); h; H; d_i)$ and $\tilde{L}(\tilde{q}(x); h; H; d_i)$ in place of $L(q(x); h; \mathcal{H}; d_i)$ and $\tilde{L}(\tilde{q}(x); h; \mathcal{H}; d_i)$ we obtain $\Delta(\lambda) = \tilde{\Delta}(\lambda)$. Hence

$$
\dot{\Delta}(\lambda_n) = \dot{\tilde{\Delta}}(\lambda_n)
$$

for all $n \in \Lambda_0$. From Lemma 2 and assumptions we get $\beta_n = \tilde{\beta}_n$. The remainder of the proof is as for Theorem 4. □

5 Conclusion

In this paper, the inverse Sturm–Liouville problems with finite number of transmission and Robin boundary conditions was studied. For this purpose, a new Hilbert space by defining a new inner product for obtaining a self–adjoint operator was defined. So, the asymptotic form of solutions, eigenvalues and eigenfunctions of this problem was obtained. Finally, we formulated the Hochestadt's result based on transformation operator for inverse Sturm–Liouville problems.

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